Creepy Phenomena — An Investigation of Viscoelasticity

Mysterious Figure #180165120

March 25th, 2019

Contents

Introduction

This report will investigate and mathematically characterise the behaviour of the viscoelastic system shown in the provided experimental video[1]. The general approach will be:

1. Relate the given volume of the syringe to the video scale
2. Track and plot the viscoelastic response of the system
3. Use the plot to calculate the coefficient of viscosity

To aid in the task of making measurements / tracking motion in the video, the open-source, freely available Tracker application was used[2]. This enabled sub-pixel measurements to be made and the frame-by-frame tracking eliminates human timing error. The data from Tracker were exported for processing in R and were subsequently integrated into the LaTeX report using Knitr.

1 Foundational Information

1.1 Viscoelastic Model & Idealisations

The video shows the Maxwell model of viscoelasticity. This is the simplest model of viscoelasticity and consists of a spring and damper (also known as a dashpot) connected in series.

[1]The video is publicly available here: `http://youtu.be/ZVK1qVkXfC4`
[2]Tracker can be found here: `https://physlets.org/tracker/`

It should be noted that this system isn't quite the ideal Maxwell model. There is also a mass component that is introducing subtle, second-order dynamics into the system. Because the mass is dropped, its momentum initially overextends the spring and a brief period of oscillation ensues. With that being said, after the oscillations subside, the system should act more-or-less like an ideal Maxwellian system. The other assumptions being made throughout this report are:

- The dashpot (damper) is ideal and obeys the equation $F = \eta\dot{x}$
- The spring is ideal (no self-damping) and obeys the equation $F = kx$
- The η and k terms in the equations above are constant
- The fibre connecting components is massless and exhibits no elasticity
- The pulley is frictionless and massless
- The internal volume of the syringe is cylindrical
- The diameter of the syringe plunger is equal to the internal diameter of the syringe
- The volume of the syringe from the first to last marking is exactly 100 ml
- The mass-hook assembly is exactly 500 g
- The acceleration of gravity acting on the mass is $9.807\,\mathrm{m\,s^{-2}}$
- The volume, mass, and acceleration of gravity are constant throughout the video
- The mass can be treated as a pure, constant force where $F = mg$
- This gravitational force is the only force acting on the system
- The syringe body and pulley assembly are fixed in space and are unyielding
- The syringe-spring-pulley assembly is perfectly level with the horizon
- Aerodynamic effects (outside of the syringe needle) are negligible

1.2 Experimental Setup

A mechanical diagram of the experimental setup can be seen in Figure 1. This system can be further simplified by replacing the pulley and weight with a constant force. This idealised take on the experimental setup perfectly represents the Maxwell model and can be seen in Figure 2.

2 Scale Calibration

2.1 A Ratio Based Calculation of Scale

Deducing scale from a video is always a difficult task and this scenario is further complicated by the lack of information provided. The volume of the syringe, however, is known and from this and the aspect ratio of the syringe, it is possible to work out the diameter of the syringe plunger. Once the diameter of the plunger is known, the scale of the video can be calculated.

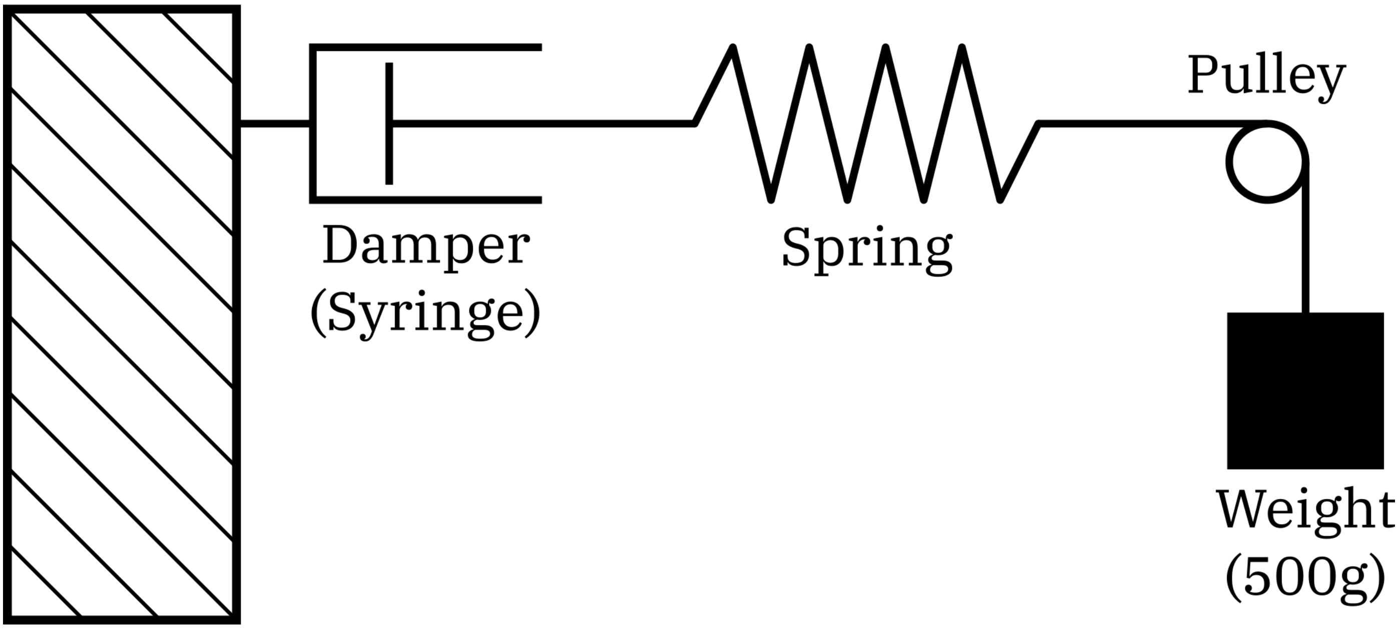


Figure 1: A mechanical diagram of the experimental setup — featuring a damper, spring, and pulley-weight assembly for force application.

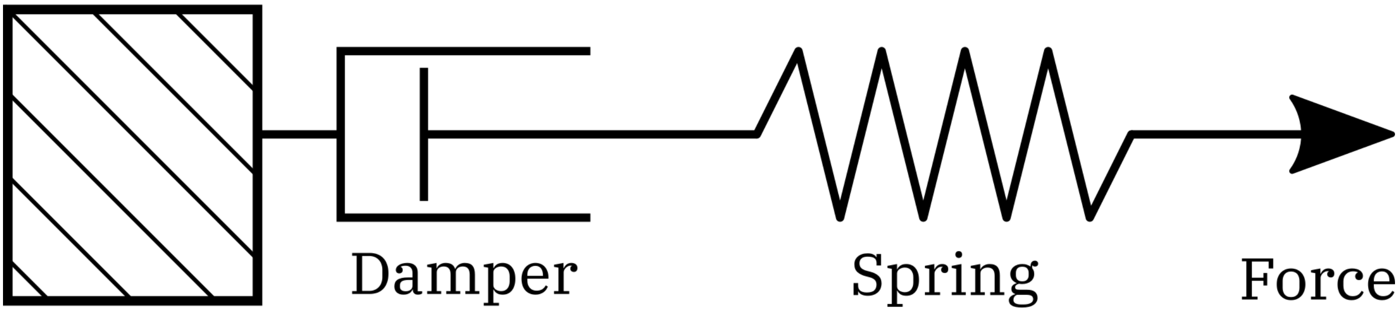


Figure 2: An idealised version of the experimental setup. This diagram is identical to that of the Maxwell model.

The first problem to overcome is that of units. We know that we will want our final units to be some measure of length. Currently, however, we only have volume. Helpfully, one millilitre is equal to one cubic centimetre[3]. Consequentially:

$$V = 100\,\mathrm{ml} = 100\,\mathrm{cm}^3$$

The next step is to find the ratio between the diameter and length of the syringe. This will allow us to rewrite the dimensions of the syringe using a single unknown which can later be solved for. Because this is a ratio, the units are irrelevant and we can measure in pixels without calibrating first. I'll define this ratio as Q.

$$Q = \frac{\text{Length}}{\text{Diameter}}$$

The error bounds are given as a standard uncertainty. This way the minimum and maximum interpretations are taken into account, but the consistency of the intermediate measurements can tighten the error bounds a little. Figure 5 in the appendix shows this measurement and Table 5c summarizes the numerical values.

[3]Fun fact, the litre is not an SI unit, however, it is among a list of units that are acceptable to use in conjunction with the SI units (`https://www.bipm.org/utils/common/pdf/si_brochure_8_en.pdf`). The correct SI unit for volume is, in fact, the *cubic metre* (m^3). Neat.

Also note that, because the original measurements contained four significant figures, all results have been rounded to four figures (though intermediate values retain their full precision).

The length from the 0 ml line to the 100 ml line was measured to be 178.4 ± 1.624 pixels and the inner diameter of the syringe — measured as the outer diameter of the plunger — was 60.62 ± 0.7577 pixels. The ratio of length to diameter was thus calculated to be 2.943 ± 0.0455.

In order to relate this ratio back to volume, it's assumed that the inside of the syringe is a perfect cylinder. The volume of the syringe is then:

$$V = \pi r^2 h$$

To put that equation in known terms, observe that the diameter (d) is double the value of r and that h, in this case, is the length (which can be expressed in terms of d and the ratio Q). Putting this all together:

$$V = \pi \left(\frac{d}{2}\right)^2 Qd$$

This can then be reduced to:

$$V = \frac{\pi}{4} Qd^3$$

Finally, solving for d:

$$d = \sqrt[3]{\frac{4V}{\pi Q}}$$

Plugging in values, I'm left with a diameter of 3.511 ± 0.0181cm. Converting to millimetres is a piece of cake, resulting in a final diameter of 35.11 ± 0.181mm. Comparing to the diameter of other, 100 ml syringes listed online by Harvard Apparatus[4] it's possible to sanity check this result. The most common diameter appears to be 34.9 mm which isn't all too far from the calculated value.[5] If we assume that the true diameter of the plunger is 34.9 mm and use that as a reference, then we can calculate the percent error using the following formula:

$$\text{Percent Error} = \frac{\text{Calculated} - \text{Reference}}{\text{Reference}} \times 100$$

Plugging in values results in an overall percent error of 0.5927 %. Less than 1% error is quite encouraging and makes it seem likely that the syringe shown in the video is of a standardized form-factor.

2.2 Calculating a Conversion Factor

As a final result, we can divide the calculated, millimetre diameter by the measured, pixel diameter in order to obtain the real size of a pixel.

$$\text{Real size of a pixel} = \frac{\text{Millimetres}}{\text{Pixels}}$$

[4]`http://www.harvardapparatus.com/media/harvard/pdf/Syringe%20Selection%20Guide.pdf`

[5]Ready to do some unethical statistics? If the wording of this sentence were slightly changed from "most common diameter" to "average diameter" — which, in everyday English, holds the same meaning — the discrete nature of syringe sizes could be conveniently overlooked. The average diameter (as opposed to mode diameter) of the syringes on the datasheet can be computed as 35.17 mm. While this value is somewhat meaningless (there are no real syringes that have a diameter of 35.17 mm), it could be used as a reference value all the same and would produce the misleadingly low percent error of −0.17 %. Don't p-hack, kids.

The tricky thing here is that that video changes scale between where initial measurements were made (when the video was zoomed in on the syringe) and where the displacement is measured (near the end of the video when the whole system is seen in motion). Because this conversion factor will be used to convert measurements made in tracking the displacement, the diameter of the syringe must be remeasured after the video zooms out to show the whole system. These secondary measurements can be seen in Figure 6c and are summarized in Table 6b. The rescaled diameter comes out to be 30.91 ± 0.1098 pixels.

Plugging this value (along with the calculated, millimetre diameter) into the equation above gives a conversion factor of 1.136 ± 0.007112 millimetres per pixel and will be used to convert all future measurements from pixels to millimetres. This is functionally equivalent to calibrating Tracker / VIEW, but allows us to propagate calibration uncertainty all the way through to the final result.

3 Analysing Viscoelastic Behaviour

3.1 Plotting the Creep Response

Once the scale of the system has been determined, the system creep must be plotted. To accomplish this, the position of the dropped mass (in pixels) was tracked frame-by-frame. The path traced by the mass can be seen in Figure 6a. Tracking the mass (as opposed to just the syringe plunger) makes it possible to see the full system dynamics (mass-spring included), but also means that swaying needs to be addressed.

While the weight hanging off of the end of the pulley was acting like a pendulum (swinging slightly from side to side), the change in angle was so minuscule that the Pythagorian distance from the pulley to the mass was nearly identical to the displacement in the y-direction. In more formal terms, if l is the true displacement, then:

$$l = \sqrt{x^2 + y^2} \quad \text{and} \quad x = y \tan\theta$$

Substitution for x results in:

$$l = \sqrt{(y \tan\theta)^2 + y^2}$$

But θ is so small, it can be considered 0:

$$\lim_{\theta \to 0} \tan\theta = 0$$

Putting it all together:

$$l = \sqrt{(y \times 0)^2 + y^2} = \sqrt{y^2} = |y|$$

Therefore, in all following calculations, the absolute value of the y-coordinate of the mass will be treated as the displacement of the mass and, consequentially, the extension of the viscoelastic system.

The data was normalised so that the first measurement was centred at (0,0) and then was plotted in Figure 3. In this figure, it is possible to see a brief, oscillatory period right after the mass is dropped. This confirms the presence of second-order dynamics and reaffirms that this system is *not* an ideal Maxwellian system.[6] After 2.5 seconds, however, the mass-spring component of the system settles and the dynamics that follow can be considered adherent to the Maxwell model.

By discarding the first 2.5 seconds of data, it's possible to plot the creep response of the dashpot component. This is shown in Figure 4. A linear regression was also done on this data and the gradient was found to be 2.873 ± 0.003879. The uncertainty for this value is the standard error of the regression coefficient. It describes the variance of the gradient if the regression were to be repeated with a different

[6] But come on, nobody is perfect!

sample. Critically, this value plays nice with our existing standard uncertainty values and can still be propagated through to the final answer. This gradient can also be thought of as a velocity with the units of *pixels per second*. We can use this velocity in the next section to calculate the viscosity coefficient η.

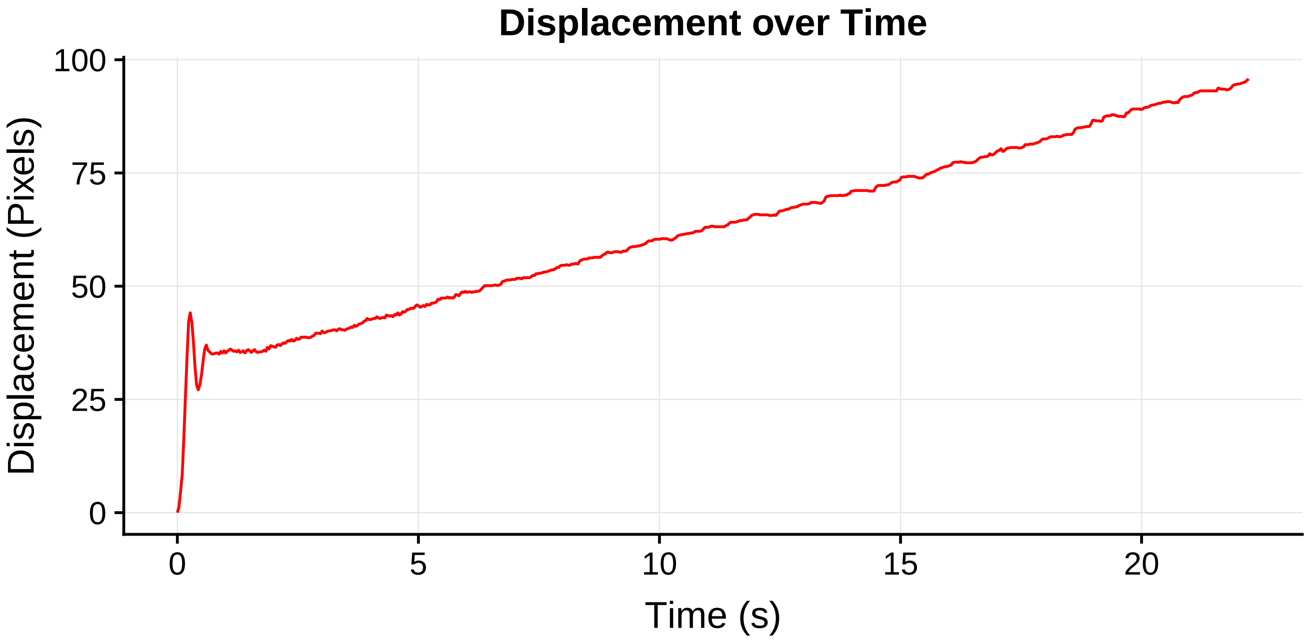


Figure 3: The full response of the viscoelastic system

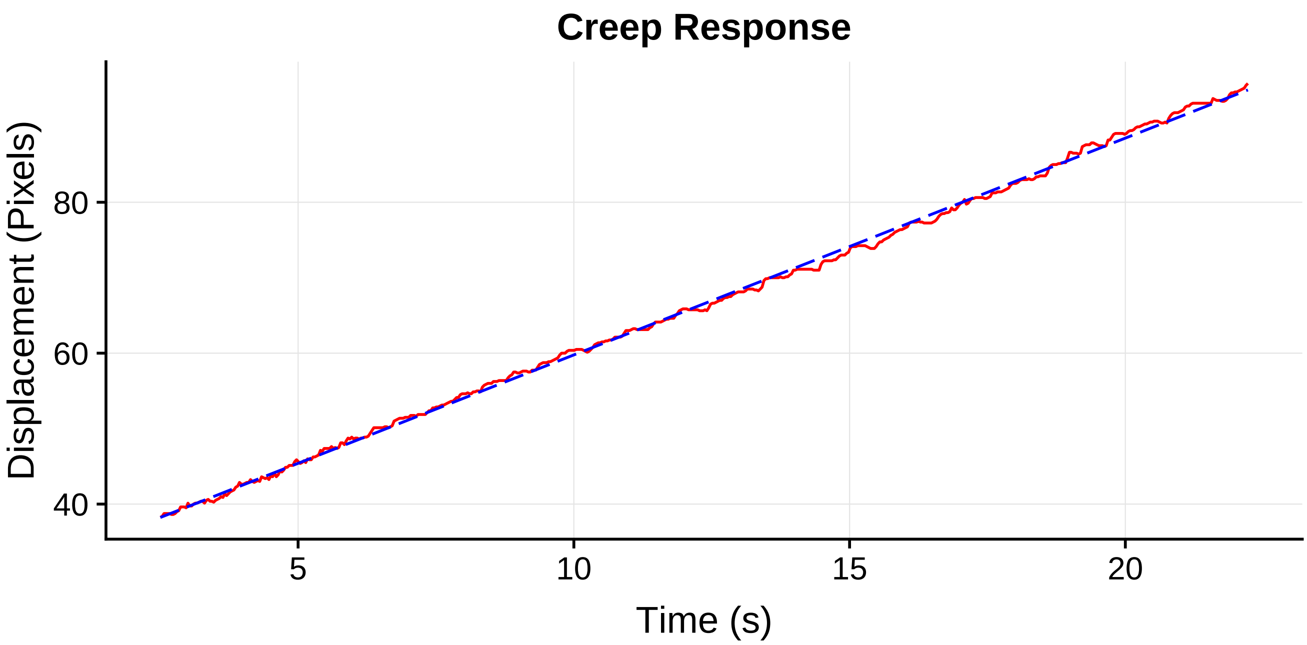


Figure 4: The cropped, idealised response of the viscoelastic system (shown in red) and the line of best fit (shown in blue).

3.2 Calculating the Viscosity Constant

Because, in the Maxwell model, the spring and damper are connected in series, several relations are immediately apparent: the forces acting on the individual components of the system must be equivalent to the overall applied force[7] and the total extension of the system must be equal to the sum of the spring and damper extensions.

$$F_d = F_s = F_g \quad \text{and} \quad x = x_d + x_s$$

Newton's second law gives the relation between force, mass, and acceleration as $F = ma$ and the acceleration of gravity at the surface of Earth is given as g; therefore:

$$F_g = mg$$

Finally, it's helpful to note down the fundamental equations for a spring and a damper:

$$F_d = \eta \dot{x}_d \quad \text{and} \quad F_s = kx_s$$

Currently, this presents three unknowns: x_d, x_s, and η (what is being solved for). The displacement relation can be used to do away with x_d and x_s. In order to avoid integrating the damper equation, the displacement relation can be differentiated and rewritten as:

$$\frac{d}{dt}x = \frac{d}{dt}(x_d + x_s) \quad \text{or} \quad \dot{x} = \dot{x}_d + \dot{x}_s$$

The fundamental equations can then be rearranged and the spring term differentiated to solve for the $\dot{x}$ values:

$$\dot{x}_d = \frac{F_d}{\eta}, \quad \dot{x}_s = \frac{d}{dt}\left(\frac{F_s}{k}\right)$$

Substituting into the displacement relation:

$$\dot{x} = \frac{F_d}{\eta} + \frac{d}{dt}\left(\frac{F_s}{k}\right)$$

As the spring term has no dependence on t, it's derivative is zero. Consequentially:

$$\dot{x} = \frac{F_d}{\eta}$$

Recall that $F_d = F_g$ and that $F_g = mg$:

$$\dot{x} = \frac{mg}{\eta}$$

Solving for η results in the following equation:

$$\eta = \frac{mg}{\dot{x}}$$

Finally it is possible to solve for η. The mass (m) is given to be 500 g, the surface acceleration of gravity (g) is $9.807\,\mathrm{m\,s^{-2}}$, and the velocity calculated earlier ($\dot{x}$) is 2.873 ± 0.003879 pixels per second. The viscosity constant η is then calculated to be 1707 ± 2.304. But wait... What are those units? Gram-metres per pixel-second? That's pretty awful.[8] Converting to SI units is possible through the following series of operations:

$$\frac{1707\,\mathrm{g \cdot m}}{\mathrm{px \cdot s}} \times \frac{\mathrm{px}}{1.136\,\mathrm{mm}} \times \frac{1000\,\mathrm{mm}}{\mathrm{m}} \times \frac{\mathrm{kg}}{1000\,\mathrm{g}}$$

[7] This can be proven by Newton's third law of motion which states every force has an equal and opposite reactionary force. The force of gravity acts on the mass, so the spring provides an opposing force, which induces the same force again in the damper, which finally transfers the force to the wall.

[8] A more professional report might say "non-standard", but I started this report with a pun, so...

It's then possible to cancel units and some common factors:

$$\frac{1707\,\mathrm{g}\cdot\cancel{\mathrm{m}}}{\cancel{\mathrm{px}}\cdot\mathrm{s}}\times\frac{\cancel{\mathrm{px}}}{1.136\,\cancel{\mathrm{mm}}}\times\frac{\cancel{1000}\,\cancel{\mathrm{mm}}}{\cancel{\mathrm{m}}}\times\frac{\mathrm{kg}}{\cancel{1000}\,\cancel{\mathrm{g}}}=\frac{1707\,\mathrm{kg}}{1.136\,\mathrm{s}}$$

At long last, we can calculate our final answer:

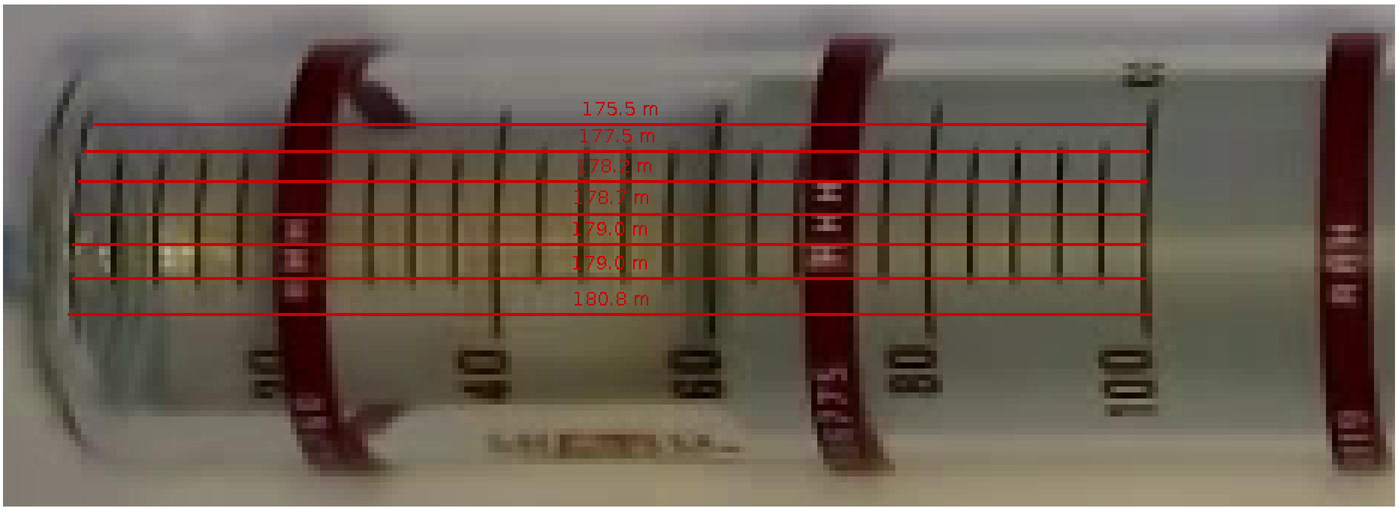
$$\boldsymbol{\eta = 1502 \pm 9.622\,\mathrm{kg\,s^{-1}}}$$

As a final, statistical hurrah, a confidence interval can be calculated from the standard uncertainty of our answer. The cumulative distribution function says that approximately 95% of data in a normal distribution (like our uncertainty distribution — see Appendix C) falls within two standard deviations from the mean. That makes it possible to say, with 95% confidence, that the true value of η lies between $1483\,\mathrm{kg\,s^{-1}}$ and $1522\,\mathrm{kg\,s^{-1}}$. With a range of only $38.49\,\mathrm{kg\,s^{-1}}$, that's not half bad![9]

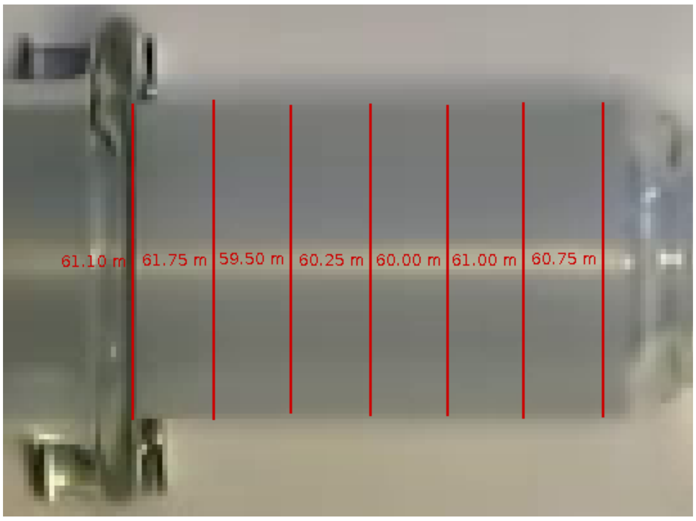
[9]It's worth noting that when working with an actual viscoelastic material, stress (σ) and strain (ϵ) would have been used instead of force (F) and displacement (x). Because stress is a force over area and the strain becomes dimensionless, the unit calculations work out a bit differently. If stress and strain values were used to calculate η, the final units would have been $\mathrm{kg\,m^{-1}\,s^{-1}}$ and the value slightly different as a result. That being said, it doesn't make much sense to think about areas of force application or fractional extensions in a discrete mechanical model like this, so using force and displacement seems justified.

Appendix

A Calibration Measurements



(a) Measuring Plunger Depth



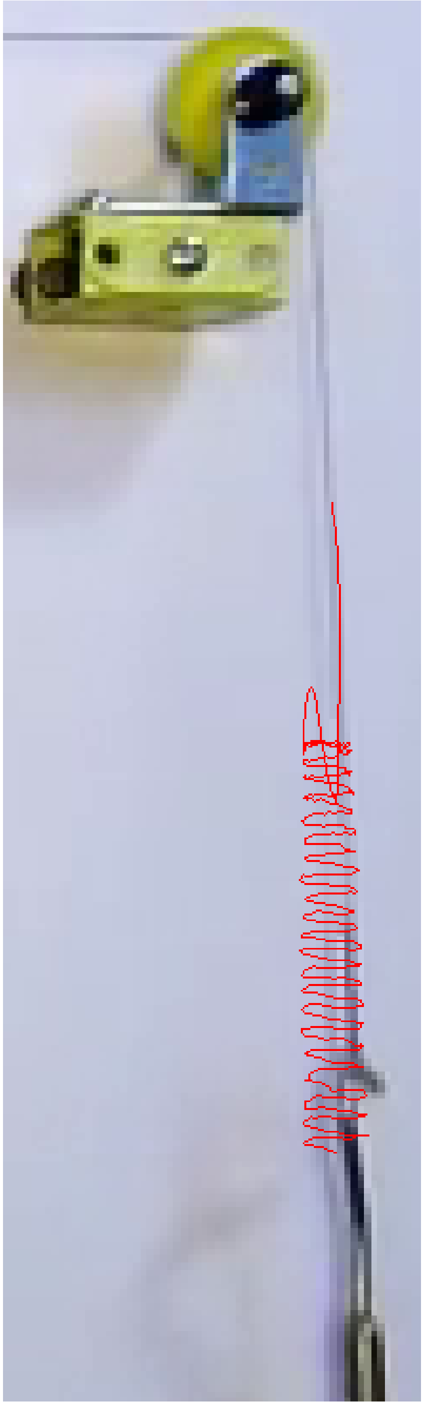
(b) Measuring Plunger Diameter

Depth (px)	Diameter (px)
175.5	61.10
177.5	61.75
178.2	59.50
178.7	60.25
179.0	60.00
179.0	61.00
180.8	60.75

(c) Numeric Values

Figure 5: Before calibration, the dimensions of the syringe must be measured in pixels so that a ratio of width to diameter can be calculated. Here the internal diameter of the syringe is found by measuring the width of the plunger. The length of the 100 ml volume is measured as the distance from the first to last black line on the syringe.

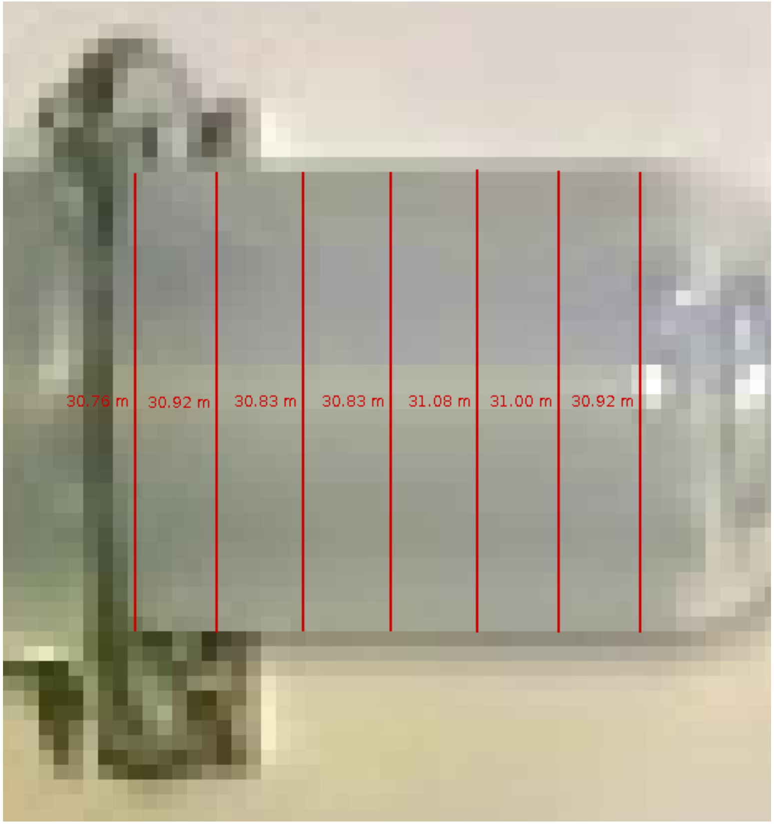
B Displacement Measurements



(a) Tracked Displacement Path

Rescaled Diameter (px)
30.76
30.92
30.83
30.83
31.08
31.00
30.92

(b) Numeric Values



(c) Remeasuring Plunger Diameter

Figure 6: The displacement of the system was tracked at the point where the hook connects the mass to the string. Since the scale of the video changed from when the original calibration measurements were taken, the diameter of the syringe plunger must be remeasured so that a correct scaling-factor can be obtained.

C Standard Uncertainty & Uncertainty Propagation

Whenever measurements are taken, there is bound to be some ambiguity. A combination of limited-precision tools and human subjectivity results in an uncertainty regarding the true value of any particular quantity. As it happens, however, the more repeat measurements that are taken, the closer and closer the average of those measurements gets to the true value.

In statistics, this phenomenon is called *The Law of Large Numbers* and the distribution of these repeat measurements is called a *Gaussian* or *Normal* distribution. The width of this distribution can then be used as a gauge for how much uncertainty is present in the measurements — this is described by the *Standard Deviation*. The standard deviation is easy to track for repeat measurements like those used in calibration or for the gradient of a regression, but how does it change when values are multiplied? Or raised to a power?

To answer this, we turn to the *Exact Formula for Propagation of Error*:

$$\sigma_x^2 = \left(\frac{\partial x}{\partial a}\right)\sigma_a^2 + \left(\frac{\partial x}{\partial b}\right)\sigma_b^2 + \cdots + \left(\frac{\partial x}{\partial n}\right)\sigma_n^2$$

The full derivation is beautiful and shows the connection between Calculus and Statistics but isn't the topic of this assignment; here, the focus will be on applying this equation.[10] Throughout this report, there have been three classes of calculations that involved uncertainty: exponentiation of uncertain values, division of uncertain values, and scaling by a definite constant. The solution of the propagation equation for each of these calculations is shown below.

C.1 Definite Scaling

$$x = ka, \quad \frac{\sigma_x}{x} = \frac{\sigma_a}{a}$$

C.2 Exponentiation

$$x = a^n, \quad \frac{\sigma_x}{x} = n\left(\frac{\sigma_a}{a}\right)$$

C.3 Multiplication & Division

$$x = \frac{a \times b}{c}, \quad \frac{\sigma_x}{x} = \sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2}$$

[10] The full derivation can be found here: `https://chem.libretexts.org/Bookshelves/Analytical_Chemistry/Supplemental_Modules_(Analytical_Chemistry)/Quantifying_Nature/Significant_Digits/Propagation_of_Error`

D R Code

D.1 Scale Calibration

```r
# Load the ratio measurements
ratio <- read.csv("../Measurements.csv")
# Round (when displaying) to 4 significant figures
options(digits = 4, knitr.digits.signif = TRUE)
# Get the average length
len <- mean(ratio$Length)
len

## [1] 178.4

# And the error
lenSD <- sd(ratio$Length)
lenSD

## [1] 1.624

# Get the average diameter
dia <- mean(ratio$Diameter)
dia

## [1] 60.62

# And the error
diaSD <- sd(ratio$Diameter)
diaSD

## [1] 0.7577

# Time to find a ratio
Q <- len / dia
Q

## [1] 2.943

# Now propagate the uncertainty (I'm calculating the standard uncertainty)
QSD <- sqrt((lenSD / len)^2 + (diaSD / dia)^2) * Q
QSD

## [1] 0.0455

# Holy mackerel, it's time for finding d!
V <- 100
d <- (4*V/(pi*Q))^(1/3)
d <- d * 10 # For millimeters
d

## [1] 35.11
```

```r
# And uncertainty, still standard uncertainty, different formula this time
dSD <- (1/3) * (QSD / Q) * d
dSD

## [1] 0.181

# Sanity checking, some values from online
exDias <- c(34.9, 34.9, 35.7) # Values found from Harvard Apparatus Datasheet
# Most common diameter
modeDia <- exDias[1]
modeDia

## [1] 34.9

# Average diameter
avgDia <- mean(exDias)
avgDia

## [1] 35.17

# Percent error in the diameter calculation (assuming mode is correct)
modeDiaError <- (d - modeDia) / modeDia * 100
modeDiaError

## [1] 0.5927

# Percent error assuming the average is correct (to make a point)
avgDiaError <- (d - avgDia) / avgDia * 100
avgDiaError

## [1] -0.17

# Get the average, rescaled diameter
rdia <- mean(ratio$RescalDia)
rdia

## [1] 30.91

# And the error
rdiaSD <- sd(ratio$RescalDia)
rdiaSD

## [1] 0.1098

# As a final, calibration result, what is the size of a pixel?
px <- d / rdia
px

## [1] 1.136

# Uncertainty again
pxSD <-  sqrt((dSD / d)^2 + (rdiaSD / rdia)^2) * px
pxSD

## [1] 0.007112
```

D.2 Analysing Viscoelastic Behaviour

```r
# Load the mass tracking data (667 data points!)
creepy <- read.csv("../DropMass.csv")
# Normalise the data
normdata <- data.frame(t = creepy$t - creepy$t[1], x = creepy$x - creepy$x[1],
y = creepy$y[1] - creepy$y)
# Trim out the first 2.5 seconds (where the oscillations are)
trimdata <- normdata[normdata$t > 2.5,]
# Time for a linear regression to get the gradient of the creep plot
lobf <- lm(y ~ t, data = trimdata)
# Extract the gradient (velocity)
vel <- coef(lobf)["t"]
vel

##     t
## 2.873

# Get the standard error of velocity
velSD <- summary(lobf)$coefficients[2,2]
velSD

## [1] 0.003879

# Put the regression values in a plottable function
reg <- function(x) vel * x + coef(lobf)["(Intercept)"]
# Here are some known constants
m <- 500 # grams
g <- 9.80665
# Calculate the viscosity coefficient
eta <- m * g / vel
eta

##    t
## 1707

# Don't forget about error!
etaSD <- (velSD / vel) * eta
etaSD

##     t
## 2.304

# Fix the units on eta
ans <- eta / px
ans

##    t
## 1502

# Final uncertainty propagation
ansSD <- sqrt((etaSD / eta)^2 + (pxSD / px)^2) * ans
ansSD

##     t
## 9.622
```