

# Under Pressure\*— Hydrostatics and Elastic Tube Distention

Mysterious Figure #180165120

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## Introduction

The goal of this report is to, using measurements from the picture found in Figure 2, calculate the elastic modulus of the experimental tubing and assess its viability as an aortal analogue. The general approach will be:

1. Work out an equation that relates elastic modulus and hydrostatic pressure
2. Plot the relationship between tube distention and pressure
3. Using the gradient of this plot, calculate the elastic modulus of the tube
4. Assess the suitability of this setup as an aortal analogue.

To aid in the task of making measurements from the picture, the open-source, freely available GIMP application was used.<sup>1</sup> This enabled precise and repeatable measurements to be made. The measurements from GIMP were exported for processing in R and were subsequently integrated into the LaTeX report using Knitr.

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\*I'm a Uni student after all ;)

<sup>1</sup>GIMP can be found here: <https://www.gimp.org/>

# 1 Relating Elastic Modulus to Hydrostatic Pressure

Let's start by enumerating the information provided in the assignment. Firstly, it is known that the hydrostatic pressure in a fluid column is given by the following equation:

$$P = \rho gh \quad (1)$$

$\rho$  is the density of the fluid,  $g$  is the acceleration of gravity at Earth's surface, and  $h$  is the height of the fluid column. Further provided are the equations for hoop stress and strain:

$$\sigma_h = P \frac{r_0}{t} \quad \text{and} \quad \epsilon_h = \frac{\Delta r}{r_0}$$

where  $r_0$  is the initial radius of the tube,  $\Delta r$  is the change in radius under pressure, and  $t$  is the thickness of the tube. Finally, the elastic modulus can be calculated as stress over strain.

$$E = \frac{\sigma_h}{\epsilon_h}$$

Putting all of this together, it is possible to write an equation for the elastic modulus of a thin-walled tube:

$$E = P \frac{r_0}{t} \times \frac{r_0}{\Delta r} = \frac{Pr_0^2}{t\Delta r}$$

While the above is the most compact representation, it would be helpful to make the relationship between  $P$  and  $\Delta r$  more explicit by separating them from the constant terms of the equation. With this in mind, the final representation of this equation will be:

$$E = \frac{P}{\Delta r} \times \frac{r_0^2}{t} \quad (2)$$

## 2 Plotting Tube Distention & Pressure

### 2.1 Finding Pressures

First things first, before the relationship between tube diameter and pressure can be plotted, the pressures in each column need to be calculated. In the assignment the column heights are given as millimetre measurements. To find pressures from these values, they can simply be converted to the SI unit of metres and be plugged in for  $h$  in Equation 1. Of course, in order to get real numbers, some assumptions need to be made about the other values, namely:

- The acceleration of gravity ( $g$ ) is to be taken as  $9.81 \text{ m s}^{-2}$
- The fluid will be assumed to be water with a density ( $\rho$ ) of  $1000 \text{ kg m}^{-3}$

Since this system is later compared to the cardiovascular system, it would, of course, be ideal to use blood as the fluid but doing so is often impractical. Seeing as the average density of blood ( $1060 \text{ kg m}^{-3}$ ) is relatively similar to that of water, the 5.66% error in density shouldn't noticeably affect the experimental findings.<sup>2</sup>

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<sup>2</sup>With the added benefit that you can work without the fear of accidentally dousing yourself with a bucket of blood.

The last thing that needs to be considered before pressures are actually calculated is measurement error and uncertainty. For the purposes of this calculation, both  $g$  and  $\rho$  will be considered as introducing negligible amounts of uncertainty.<sup>3</sup> The value that introduces a more meaningful amount of uncertainty is the height of the fluid in the column ( $h$ ).

It seems reasonable to assume that this quantity was measured using some sort of ruler (as the measurements are given in whole millimetres) and therefore it's possible to measure a value that is a millimetre or so off. For the purpose of these calculations then, the error in  $h$  will be taken as  $\pm 1\text{mm}$ .

The last trick here is to convert that absolute uncertainty into *standard uncertainty* which is represented as a standard error. Since  $\pm 1\text{mm}$  is considered to be a range that (hopefully) all true measurements would fall into, and nearly all (99.7%) of values lie within three standard deviations from the mean, the standard uncertainty would be 0.333. Since this is a formalised statistical framework for tracking and propagating error, while all numbers will be given with uncertainty values, the maths will be done mostly transparently and in the background throughout the report. More detail about error tracking can be found in Appendix B and the background maths can be seen in the R code of Appendix C.

Finally, it is possible to calculate the pressures for each column height (as shown in Table 1).

Table 1: Column heights and their corresponding pressures. Note that the column heights in the table are only present as labels of the image sections (hence the absence of an uncertainty value).

Column Height (mm)	Pressure (Pa)
0	0
72	706 $\pm$ 3.27
128	1260 $\pm$ 3.27
169	1660 $\pm$ 3.27
226	2220 $\pm$ 3.27
293	2870 $\pm$ 3.27
350	3430 $\pm$ 3.27
395	3870 $\pm$ 3.27

## 2.2 Measuring Diameters

Now that pressures have been calculated, the diameters need to be measured. This was done in GIMP using the “Measure” tool and seven measurements were made per section, 15 pixels apart. The raw values of all of these measurements, their averages, and standard uncertainties, in addition to the source image, can be found in Appendix A.

Luckily, since it was possible to make repeat measurements of the diameter, it's trivial to calculate the uncertainty as the standard error of this data. The last tricky thing is scaling from pixels to millimetres (and eventually metres). The assignment gives the diameter of the tube (when there is 0 pressure) as 20 mm. Because this measurement is still made in whole millimetres, it will be assumed that it was measured in a similar way as the column height and it will be assigned a  $\pm 1\text{mm}$  absolute uncertainty making the standard uncertainty — 0.333 mm — the same as before.

Because everything will eventually need to be converted to metres, it would be helpful to define a conversion factor from pixels to metres. This value will be defined as follows:

$$\frac{\text{Given Initial Diameter (mm)}}{\text{Measured Initial Diameter (px)}} \times \frac{1\text{m}}{1000\text{mm}}$$

<sup>3</sup>To be truly pedantic, this assumes this experiment was performed on earth, outside of a centrifuge, and that the “water” was composed of two <sup>1</sup>H atoms and one <sup>16</sup>O atom — as opposed to some exotic deuterium (<sup>2</sup>H) forms of water... It's somewhere around here that things start to get silly.

Plugging in some numbers, it's possible to work out a value.

$$\frac{20 \pm 0.333\text{mm}}{230 \pm 0.369\text{px}} \times \frac{1\text{m}}{1000\text{mm}} = 8.71 \times 10^{-5} \pm 1.46 \times 10^{-6}\text{m px}^{-1}$$

The final diameters, converted to millimetres, are visible in Table 2.

Table 2: Column heights and diameters. The column heights, as before, are only present as labels for the image sections.

Column Height (mm)	Diameter (mm)
0	$20.0 \pm 0.336$
72	$20.3 \pm 0.342$
128	$20.7 \pm 0.348$
169	$21.0 \pm 0.352$
226	$21.2 \pm 0.357$
293	$21.8 \pm 0.366$
350	$22.1 \pm 0.371$
395	$22.6 \pm 0.381$

Note that, in the background and in all subsequent plots, the change in diameter is in the units of metres. Table 2 contains millimetres only because the values are a bit easier on the eyes.

### 2.3 The Final Plot

Finally, with all numbers gathered, uncertainties tracked, and units converted, it's time to plot the relationship between pressure and distension. The final graph can be seen in Figure 1.

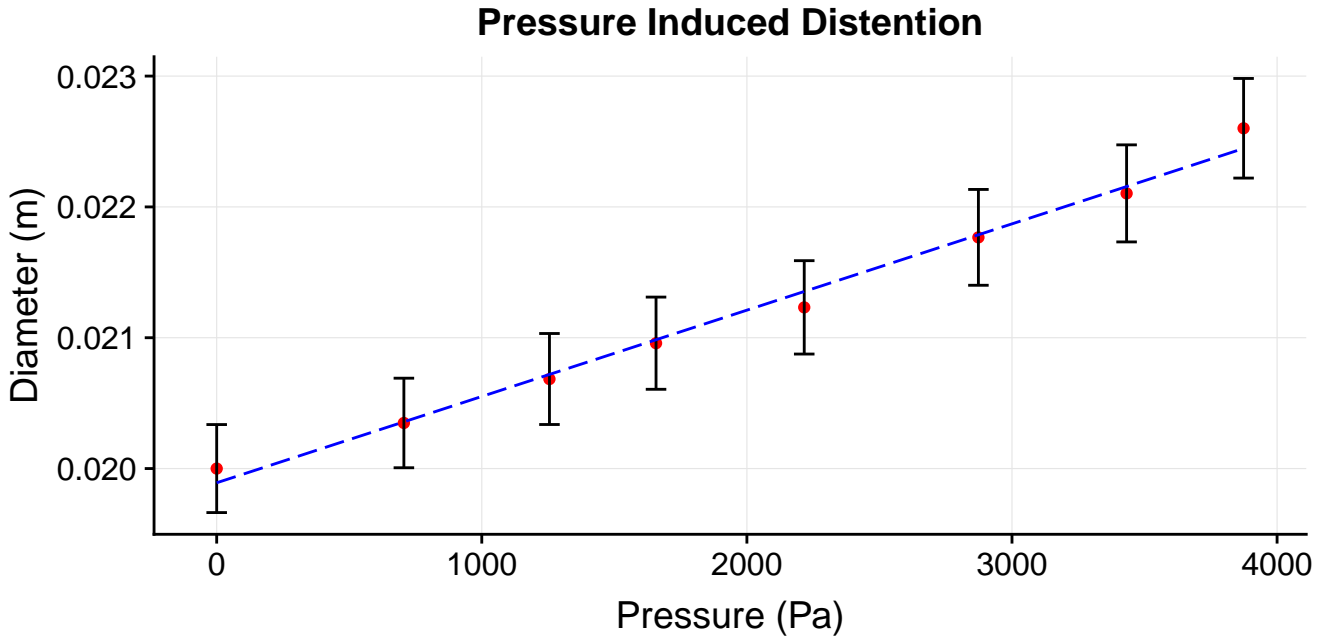


Figure 1: The diameters at various pressures (shown in red) and the line of best fit (shown in blue). Error bars show the standard uncertainty of the measured diameters. Horizontal error bars were added to show uncertainty in pressure measurements, but were removed because they were too small to visualise.

### 3 Calculating the Elastic Modulus

The relevant parameter from the plot in the previous section is the gradient. That is the value that relates the pressure and diameter. More specifically, the gradient above shows:

$$\frac{\Delta D}{\Delta P} \quad \text{which looks similar to the variable} \quad \frac{P}{\Delta r} \quad \text{term from equation 2}$$

Note that, in this context,  $D$  is the tube diameter. Rearranging  $\frac{\Delta D}{\Delta P} = G$  to resemble  $\frac{P}{\Delta r} = \dots$  will allow the gradient ( $G$ ) to be plugged into Equation 2 to solve for the elastic modulus.

$$\begin{aligned} \Delta P &= P - P_0 \quad \text{and} \quad P_0 = 0 \quad \text{so} \quad \Delta P = P \\ \frac{\Delta D}{P} &= G \quad \text{and} \quad \Delta D = 2\Delta r \quad \text{so} \quad \frac{2\Delta r}{P} = G \\ \frac{\Delta r}{P} &= \frac{G}{2} \\ \frac{P}{\Delta r} &= \frac{2}{G} \end{aligned}$$

Substituting this into Equation 2 will give the final equation for the elastic modulus:

$$E = \frac{2}{G} \times \frac{r_0^2}{t}$$

Regarding the values of the constant terms,  $r_0$  is known from earlier calculations to simply be half of the initial diameter and the tube thickness ( $t$ ) is given to be 1 mm. Seeing as the tube thickness is a very small value, it was likely measured with very precise equipment. This, in concert with the fact that other, larger uncertain terms are present in the equation, means that  $t$  will be treated as a certain value for the purposes of this calculation.

Turning towards the variable term, the gradient comes out to be  $6.6 \times 10^{-7} \pm 2.7 \times 10^{-8} \text{ m Pa}^{-1}$ . The uncertainty term here comes from the standard error of the gradient as given by the regression. This value encodes the *variation in the gradient* if the regression were to be run on a different sample of data and plays nice with the mathematical framework already established.

Without further ado:

$$E = \frac{2 \text{ Pa}}{6.6 \times 10^{-7} \pm 2.7 \times 10^{-8} \text{ m}} \times \frac{(0.01 \pm 1.67 \times 10^{-4})^2 \text{ m}^2}{0.001 \text{ m}} = \mathbf{3.03 \times 10^5 \pm 1.6 \times 10^4 \text{ Pa}}$$

#### 3.1 Confidence In Calculation

This value seems reasonable, despite the somewhat large numbers involved. The fact of the matter is that Pascals are small units and most elastic moduli are measured in Gigapascals (GPa) or sometimes Megapascals (MPa). Comparing to some other values<sup>4</sup> shows that the calculated modulus is actually quite small for normal materials, but seems reasonable for an elastic tube. Converting things to MPa, some comparisons can be made.

$$E = 0.303 \pm 0.016 \text{ MPa}$$

The paper by Bernal M et al.<sup>5</sup> looks at the elastic moduli of pig carotid arteries as well as the synthetic analogue urethane. It's quite possible that the tube used in this experiment was also a

<sup>4</sup>[https://www.engineeringtoolbox.com/young-modulus-d\\_417.html](https://www.engineeringtoolbox.com/young-modulus-d_417.html)

<sup>5</sup><https://www.ncbi.nlm.nih.gov/pmc/articles/PMC3129600/>

urethane tube as the paper gives it an elastic modulus of roughly 0.3 MPa which comes quite close to the value calculated here.

Since standard uncertainty plays nicely with many statistical tools, it is rather easy to calculate a confidence interval. Recall that two standard deviations from the mean (or two standard uncertainties from the value, in this case) encompasses 95% of data. Therefore, forming a 95% confidence interval requires doubling the uncertainty and adding and subtracting it from the mean value. It is possible to claim then, with 95% confidence, that the true value of the elastic modulus lies somewhere between 0.271 MPa and 0.335 MPa.

It's certainly not a perfect estimate, but overall, it seems serviceable. Different materials regularly differ in elastic modulus by orders of magnitude, so a range of only 0.064 MPa is pretty good.

## 4 Assessing Suitability as an Aortic Analogue

To assess the viability of this setup as a model for the aorta, the aortic and model distentions over the same changes in pressure can be compared. For the aorta, it's said that diameter changes by 10% between 80mmHg and 120mmHg. Because the question explicitly asks this case to be considered, these numbers will be left "as is" without any attached uncertainties.

Starting with the obvious, units need to be converted to Pa before they can be plugged into the model. The conversions are:

$$80 \text{ mmHg} \times \frac{133 \text{ Pa}}{1 \text{ mmHg}} = 1.07 \times 10^4 \text{ Pa}$$

$$120 \text{ mmHg} \times \frac{133 \text{ Pa}}{1 \text{ mmHg}} = 1.6 \times 10^4 \text{ Pa}$$

These pressures can easily be plugged into the previous regression in order to predict the distention of the model. Since it hasn't been mentioned elsewhere, it's worth pointing out that the y-intercept of the regression is  $0.0199 \pm 6.38 \times 10^{-5} \text{ m}$ . Note that the following calculations still take uncertainties into account, but the  $\pm$  bits on the left are omitted for clarity.

$$\text{Regression Gradient} \times \text{Pressure} + \text{Regression Intercept} = \text{Predicted Diameter}$$

$$(6.6 \times 10^{-7} \text{ mPa}^{-1} \times 1.07 \times 10^4 \text{ Pa}) + 0.0199 \text{ m} = \mathbf{0.0269 \pm 0.0011 \text{ m}}$$

$$(6.6 \times 10^{-7} \text{ mPa}^{-1} \times 1.6 \times 10^4 \text{ Pa}) + 0.0199 \text{ m} = \mathbf{0.0304 \pm 0.00125 \text{ m}}$$

Unfortunately, there is some ambiguity in which value to calculate next here. There is a 10% change between the two pressures, but it's not specified if that is a 10% increase or 10% decrease. To cover all bases, both percentages will be calculated. Again, the  $\pm$  bits have been omitted for clarity.

$$\% \text{-Increase} = \frac{0.0304 - 0.0269}{0.0269} = 13.1 \pm 6.2\%$$

$$\% \text{-Decrease} = \frac{0.0304 - 0.0269}{0.0304} = 11.6 \pm 5.48\%$$

Comparing either of these values to the change seen in the aorta is rather promising. While both of these percentages are a little high, the aortic 10% safely lies inside their large error bounds. It is important to note that this only shows that the aorta and this model have similar distention-pressure relations and makes no claim to the similarity of the elastic moduli of the aorta and model tube material. With that being said, when it comes to fractional distention, this setup is certainly a suitable model of the aorta.

# Appendix

## A Source Image & Measurements

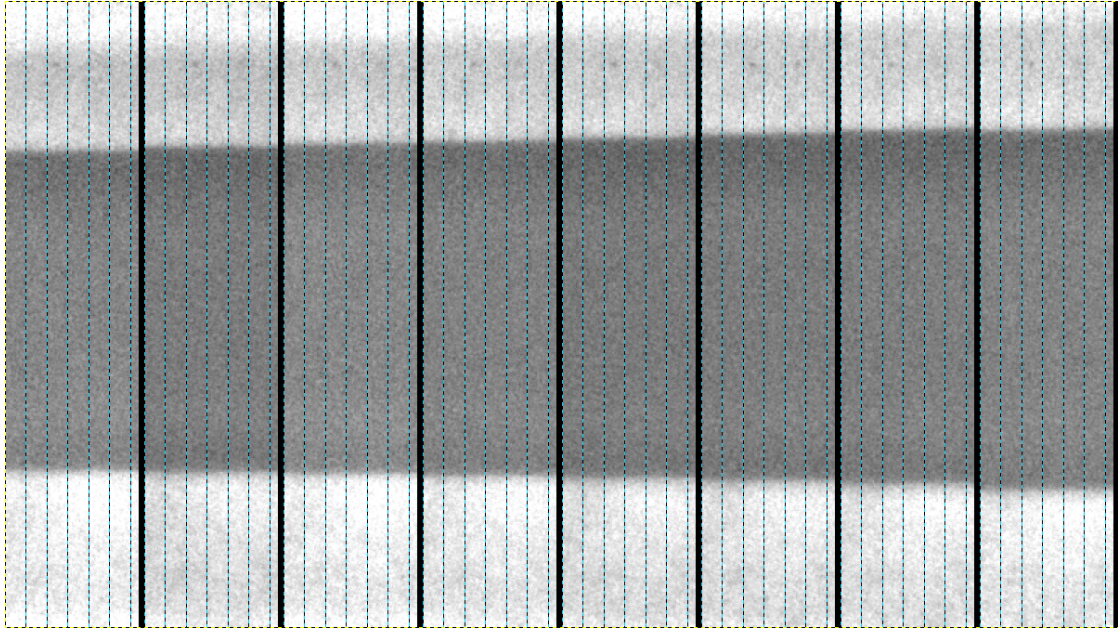


Figure 2: Experimental image from which all measurements were made. It should be noted that this version of the image has had its contrast slightly boosted and vertical guides (shown as blue, dotted lines) have been added to show where diameter measurements were taken.

Table 3: Raw diameter measurements in pixels for each fluid level. Seven measurements were taken per section. The average diameters, as well as their standard uncertainties, can be seen in the last two rows of the table.

	Fluid Level (mm)							
	0mm	72mm	128mm	169mm	226mm	293mm	350mm	395mm
Meas. 1	231	233	236	240.	243	249	253	257
Meas. 2	230.	235	237	240.	244	249	253	259
Meas. 3	229	234	238	241	244	249	254	259
Meas. 4	229	232	237	239	242	250.	254	260.
Meas. 5	228	233	237	241	244	250.	253	260.
Meas. 6	230.	234	239	242	244	250.	254	261
Meas. 7	230.	234	238	241	245	252	255	260.
Mean	230.	234	237	241	244	250.	254	259
SU	0.369	0.369	0.369	0.369	0.360	0.404	0.286	0.481

## B Standard Uncertainty & Uncertainty Propagation

Whenever measurements are taken, there is bound to be some ambiguity. A combination of limited-precision tools and human subjectivity results in an uncertainty regarding the true value of any particular quantity. As it happens, however, the more repeat measurements that are taken, the closer and closer the average of those measurements gets to the true value.<sup>6</sup>

In statistics, this phenomenon is called *The Law of Large Numbers* and the distribution of these repeat measurements is called a *Gaussian* or *Normal* distribution. The width of this distribution can then be used as a gauge for how much uncertainty is present in the measurements — this is described by the *standard uncertainty*. The standard uncertainty is related to the standard deviation (it's actually another name for standard error) and is easy to track for repeat measurements like those used in calibration or for the gradient of a regression, but how does it change when values are multiplied? Or raised to a power?

To answer this, one turns to the *Exact Formula for Propagation of Error*:

$$\sigma_x^2 = \left(\frac{\partial x}{\partial a}\right) \sigma_a^2 + \left(\frac{\partial x}{\partial b}\right) \sigma_b^2 + \cdots + \left(\frac{\partial x}{\partial n}\right) \sigma_n^2$$

The full derivation is beautiful and shows the connection between Calculus and Statistics but isn't the topic of this assignment; here, the focus will be on applying this equation.<sup>7</sup> Throughout this report, there have been four classes of calculations that involved uncertainty: addition and subtraction of uncertain values, exponentiation of uncertain values, multiplication and division of uncertain values, and scaling by a definite constant. The solution of the propagation equation for each of these calculations is shown below.

### B.1 Addition & Subtraction

$$x = a + b - c, \quad \sigma_x = \sqrt{\sigma_a^2 + \sigma_b^2 + \sigma_c^2}$$

### B.2 Definite Scaling

$$x = ka, \quad \frac{\sigma_x}{x} = \frac{\sigma_a}{a}$$

### B.3 Exponentiation

$$x = a^n, \quad \frac{\sigma_x}{x} = n \left(\frac{\sigma_a}{a}\right)$$

### B.4 Multiplication & Division

$$x = \frac{a \times b}{c}, \quad \frac{\sigma_x}{x} = \sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2}$$

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<sup>6</sup>On an uncharacteristically serious note, this is a wonderful resource on the topic of measurement uncertainty: <https://www.dit.ie/media/physics/documents/GPG11.pdf>

<sup>7</sup>The full derivation can be found here: [https://chem.libretexts.org/Bookshelves/Analytical\\_Chemistry/Supplemental\\_Modules\\_\(Analytical\\_Chemistry\)/Quantifying\\_Nature/Significant\\_Digits/Propagation\\_of\\_Error](https://chem.libretexts.org/Bookshelves/Analytical_Chemistry/Supplemental_Modules_(Analytical_Chemistry)/Quantifying_Nature/Significant_Digits/Propagation_of_Error)



## C R Code

```
# Define some constants that will be useful later
t <- 1 / 1000 # m
hg2pa <- 133.322 # Pa / mmHg
g <- 9.80665 # m/s^2
rho <- 1000 # kg/m^3
blood <- 1060 # kg/m^3
# What is the percent error between blood and water densities?
bloodDiff <- (blood - rho) / blood * 100
# Load the diameter measurements
meas <- read.csv("../Diameters.csv")
# Extract the column heights from the header
levels <- unlist(map(colnames(meas), (function (str) strtoi(substring(str,2))))))
# Give them an uncertainty
levelsSU <- unlist(map(levels, function (x) 1/3))
# Give our rows nice names
rownames(meas) <- paste("Meas.",seq(1,7))
# Give our columns nice names
colnames(meas) <- paste0(levels, "mm")
# Convert levels to metres
h <- levels / 1000
# Propagate / scale error
hSU <- h * levelsSU / levels
# Find pressures
p <- rho * g * h
pSU <- p * hSU / h
# Find the average measured diameters for each segment
dias <- colMeans(meas)
# Find the standard uncertainty for each segment
diasSU <- apply(meas,2,sd) / sqrt(nrow(meas))
# The initial tube diameter is given to be 20mm
d <- 20 # mm
# Assume again a millimetre error margin (divided by 3 for standard uncertainty)
dSU <- 1/3 # mm
# Let's find a conversion factor (from px to mm)
mmpx <- d / dias[1]
# Two uncertain values here, so new error formula
mmpxSU <- mmpx * sqrt((dSU/d)^2 + (diasSU[1]/dias[1])^2)
# Scale to metres
px2m <- mmpx / 1000
px2mSU <- px2m * mmpxSU / mmpx
# Finally convert to metres
finDias <- px2m * dias
finDiasSU <- finDias * sqrt((px2mSU/px2m)^2 + (diasSU/dias)^2)
# Plot data
plotData <- data.frame(x = p, xSU = pSU, y = finDias, ySU = finDiasSU)
# Perform a linear regression on the data
lobf <- lm(y ~ x, data = plotData)
# Extract the gradient
```

```

grad <- coef(lobf)["x"]
# Get the standard error / uncertainty of the gradient
gradSU <- summary(lobf)$coefficients[2,2]
# Extract the intercept
int <- coef(lobf)["(Intercept)"]
# Get the standard error / uncertainty of the intercept
intSU <- summary(lobf)$coefficients[1,2]
# Put the regression values in a plotable function
reg <- function(x) grad * x + coef(lobf)["(Intercept)"]
# Find r_0 from d
r0 <- d / 2 / 1000
r0SU <- r0 * dSU / d
# r0^2 is:
r02 <- r0 ^ 2
# New error formula for this one
r02SU <- 2 * r02 * r0SU / r0
# r_0^2 / G
rg <- r02 / grad
# Error, last new formula
rgSU <- rg * sqrt((gradSU / grad)^2 + (r02SU / r02)^2)
# Finally scale by 2 / t
E <- rg * 2 / t
ESU <- E * rgSU / rg
# Having things in MPa would be nice
EM <- E / 1000000
EMSU <- EM * ESU / E
# Comparing to the aorta, let's find mmHg measurements in Pa
lowAo <- 80 * hg2pa
highAo <- 120 * hg2pa
# Predict some distentions from those pressures
pDiaLow <- reg(lowAo)
pDiaHigh <- reg(highAo)
pDiaLowSU <- sqrt((pDiaLow * gradSU / grad)^2 + intSU^2)
pDiaHighSU <- sqrt((pDiaHigh * gradSU / grad)^2 + intSU^2)
# Start with the common parts of the growth and shrink percents
change <- pDiaHigh - pDiaLow
changeSU <- sqrt(pDiaHighSU^2 + pDiaLowSU^2)
# Scale up by 100 to get a percent between 0 and 100
pchange <- 100 * change
pchangeSU <- pchange * changeSU / change
# Percent increase from 80mmHg
grow <- pchange / pDiaLow
growSU <- grow * sqrt((pchangeSU / pchange)^2 + (pDiaLowSU / pDiaLow)^2)
# And percent decrease from 120mmHg
shrink <- pchange / pDiaHigh
shrinkSU <- shrink * sqrt((pchangeSU / pchange)^2 + (pDiaHighSU / pDiaHigh)^2)

```